Anomalies Revisited

Edward Witten

Lecture At Strings 2015
ICTS-TITR, Bangalore, June 22, 2015
Let me start with this question:
Let me start with this question: In string perturbation theory, how does one decide if a given string theory compactification is modular-invariant?
Let me start with this question: In string perturbation theory, how does one decide if a given string theory compactification is modular-invariant? Actually I want to narrow the question to the case that the anomalies of interest all involve fermions (possibly fermions coupled as part of a $\sigma$-model).
Let me start with this question: In string perturbation theory, how does one decide if a given string theory compactification is modular-invariant? Actually I want to narrow the question to the case that the anomalies of interest all involve fermions (possibly fermions coupled as part of a $\sigma$-model). That is because I want to focus on questions that have analogs in higher dimension – for fermions that live on the world-volume of a brane, and in other applications of quantum field theory above $1+1$ dimension.
Let me start with this question: In string perturbation theory, how does one decide if a given string theory compactification is modular-invariant? Actually I want to narrow the question to the case that the anomalies of interest all involve fermions (possibly fermions coupled as part of a $\sigma$-model). That is because I want to focus on questions that have analogs in higher dimension – for fermions that live on the world-volume of a brane, and in other applications of quantum field theory above $1+1$ dimension. In two dimensions, we could study more general questions of rational conformal field theory, but we are not going to discuss things that would be special to two dimensions.
The most familiar answer is:

To decide if the theory is modular-invariant, one calculates the partition function $Z(\tau)$ and then one checks whether it is modular-invariant

$$Z(\tau) = Z(\tau + 1) = Z\left(-\frac{1}{\tau}\right).$$
The most familiar answer is: To decide if the theory is modular-invariant, one calculates the partition function $Z(\tau)$. 
The most familiar answer is: To decide if the theory is modular-invariant, one calculates the partition function $Z(\tau)$ and then one checks whether it is modular-invariant

$$Z(\tau) = Z(\tau + 1) = Z(-1/\tau).$$
The most familiar answer is: To decide if the theory is modular-invariant, one calculates the partition function $Z(\tau)$ and then one checks whether it is modular-invariant

$$Z(\tau) = Z(\tau + 1) = Z(-1/\tau).$$
This answer is certainly correct as far as it goes, but it has a few drawbacks.
This answer is certainly correct as far as it goes, but it has a few drawbacks. We might not be able to compute the complete $Z(\tau)$ (especially in higher genus or higher dimension, or if the fermions are part of a $\sigma$-model).
This answer is certainly correct as far as it goes, but it has a few drawbacks. We might not be able to compute the complete $Z(\tau)$ (especially in higher genus or higher dimension, of if the fermions are part of a $\sigma$-model). Even if we can, the result might be very complicated and one might wonder if the anomaly, which is something simple, can somehow be computed directly rather than first computing the complete function $Z(\tau)$. 
This answer is certainly correct as far as it goes, but it has a few drawbacks. We might not be able to compute the complete $Z(\tau)$ (especially in higher genus or higher dimension, or if the fermions are part of a $\sigma$-model). Even if we can, the result might be very complicated and one might wonder if the anomaly, which is something simple, can somehow be computed directly rather than first computing the complete function $Z(\tau)$. Shouldn’t it be easier to decide if a theory is anomalous than to solve the theory?
Beyond this, there is something a little misleading about the standard answer: “Compute the partition function and check if it is modular-invariant.”
Beyond this, there is something a little misleading about the standard answer: “Compute the partition function and check if it is modular-invariant.” This answer presumes that one knows how to compute the partition function.
Beyond this, there is something a little misleading about the standard answer: “Compute the partition function and check if it is modular-invariant.” This answer presumes that one knows how to compute the partition function. That is true in genus 1 because the partition function can be computed in a Hamiltonian formalism as a sum over states,

\[ Z(\tau) = \text{Tr} \exp(-\beta H + i\theta P). \]
In higher genus (or on a generic manifold in higher dimension), there is (well, almost) no standard and well-known recipe to calculate the path integral for chiral fermions.

That is because, in the case of chiral fermions, it is not clear how to define the phase of the fermion measure.
We often call the fermion path integral a “determinant” or a “Pfaffian,” but this is a term of art.
We often call the fermion path integral a “determinant” or a “Pfaffian,” but this is a term of art. If a “determinant” is supposed to be a regularized product of eigenvalues, then the path integral for chiral fermions is not really a determinant in that specific sense,
We often call the fermion path integral a “determinant” or a “Pfaffian,” but this is a term of art. If a “determinant” is supposed to be a regularized product of eigenvalues, then the path integral for chiral fermions is not really a determinant in that specific sense, because the fermion eigenvalue problem

\[ i\dot{\psi} = \lambda \psi \]

only makes sense if both chiralities of fermion are present.
We often call the fermion path integral a “determinant” or a “Pfaffian,” but this is a term of art. If a “determinant” is supposed to be a regularized product of eigenvalues, then the path integral for chiral fermions is not really a determinant in that specific sense, because the fermion eigenvalue problem

\[ i\hat{D}\psi = \lambda\psi \]

only makes sense if both chiralities of fermion are present. The absolute value of the fermion path integral \( Z_\psi \) can be defined as a regularized product of eigenvalues, but not \( Z_\psi \) itself.
What then do we mean by the fermion path integral?
What then do we mean by the fermion path integral? In a sense, giving the best simple answer I can to this question is the purpose of the lecture.
What then do we mean by the fermion path integral? In a sense, giving the best simple answer I can to this question is the purpose of the lecture. But anyway, let us start with the best answer that is well known.
I should say that in some particular cases, ad hoc answers can be given to the sort of questions I will be discussing today.
I should say that in some particular cases, ad hoc answers can be given to the sort of questions I will be discussing today. There won’t be time to analyze all the ad hoc answers and see how far they would get us.
I should say that in some particular cases, ad hoc answers can be given to the sort of questions I will be discussing today. There won’t be time to analyze all the ad hoc answers and see how far they would get us. As one example, two-dimensional topology is so simple that in that particular case one could normalize the higher genus amplitudes using factorization and I imagine that is the first answer most people would give.
Although there is no standard method to define the fermion path integral $Z_\psi$, there is a standard method to compute how it is supposed to change when one varies some background field – for example, the metric tensor or gauge fields that the fermions are coupled to.
Although there is no standard method to define the fermion path integral $Z_\psi$, there is a standard method to compute how it is supposed to change when one varies some background field – for example, the metric tensor or gauge fields that the fermions are coupled to. Consider fermions on some $D$-manifold $M$ – which would be the string worldsheet in the example that I started with. (The fermions may also be coupled to gauge fields but for brevity I emphasize the coupling to gravity.)
Although there is no standard method to define the fermion path integral $Z_\psi$, there is a standard method to compute how it is supposed to change when one varies some background field – for example, the metric tensor or gauge fields that the fermions are coupled to. Consider fermions on some $D$-manifold $M$ – which would be the string worldsheet in the example that I started with. (The fermions may also be coupled to gauge fields but for brevity I emphasize the coupling to gravity.) The response of $Z_\psi$ to a change in the metric of $M$ is given by a standard formula

$$\frac{\delta}{\delta g_{\mu\nu}(x)} \log Z_\psi = \langle T_{\mu\nu}(x) \rangle.$$
Although there is no standard method to define the fermion path integral $Z_\psi$, there is a standard method to compute how it is supposed to change when one varies some background field – for example, the metric tensor or gauge fields that the fermions are coupled to. Consider fermions on some $D$-manifold $M$ – which would be the string worldsheet in the example that I started with. (The fermions may also be coupled to gauge fields but for brevity I emphasize the coupling to gravity.) The response of $Z_\psi$ to a change in the metric of $M$ is given by a standard formula

$$\frac{\delta}{\delta g_{\mu\nu}(x)} \log Z_\psi = \langle T_{\mu\nu}(x) \rangle.$$ 

The right hand side is well-defined in a theory that is free of perturbative anomalies.
Although there is no standard method to define the fermion path integral $Z_\psi$, there is a standard method to compute how it is supposed to change when one varies some background field – for example, the metric tensor or gauge fields that the fermions are coupled to. Consider fermions on some $D$-manifold $M$ – which would be the string worldsheet in the example that I started with. (The fermions may also be coupled to gauge fields but for brevity I emphasize the coupling to gravity.) The response of $Z_\psi$ to a change in the metric of $M$ is given by a standard formula

$$\frac{\delta}{\delta g_{\mu\nu}(x)} \log Z_\psi = \langle T_{\mu\nu}(x) \rangle.$$  

The right hand side is well-defined in a theory that is free of perturbative anomalies. For instance, in perturbation theory we can just compute it from Feynman diagrams.
Although there is no standard method to define the fermion path integral $Z_\psi$, there is a standard method to compute how it is supposed to change when one varies some background field – for example, the metric tensor or gauge fields that the fermions are coupled to. Consider fermions on some $D$-manifold $M$ – which would be the string worldsheet in the example that I started with. (The fermions may also be coupled to gauge fields but for brevity I emphasize the coupling to gravity.) The response of $Z_\psi$ to a change in the metric of $M$ is given by a standard formula

$$\frac{\delta}{\delta g^{\mu\nu}(x)} \log Z_\psi = \langle T_{\mu\nu}(x) \rangle.$$ 

The right hand side is well-defined in a theory that is free of perturbative anomalies. For instance, in perturbation theory we can just compute it from Feynman diagrams. There is no subtlety in the normalization.
There is a possible problem: when we compute $\langle T_{\mu\nu}(x) \rangle$ we might run into perturbative anomalies.

Specifically, we would like $Z_\psi$ to be invariant under infinitesimal diffeomorphisms of $M$, $\delta x^\mu = v^\mu(x)$. The condition for this is conservation of the stress tensor $D_\mu \langle T_{\mu\nu}(x) \rangle = 0$.

In some theories, we cannot regularize $\langle T_{\mu\nu}(x) \rangle$ in such a way that this condition will be satisfied. We say that those theories have perturbative gravitational anomalies and we discard them. At least in the traditional view, we only study more subtle questions like modular invariance if the perturbative anomalies cancel (after possibly combining together the contributions of a variety of different bose and fermi fields).
There is a possible problem: when we compute $\langle T_{\mu\nu}(x) \rangle$ we might run into perturbative anomalies. Specifically, we would like $Z_\psi$ to be invariant under infinitesimal diffeomorphisms of $M$, 

$$\delta x^\mu = \nu^\mu(x).$$
There is a possible problem: when we compute $\langle T_{\mu\nu}(x) \rangle$ we might run into perturbative anomalies. Specifically, we would like $Z_\psi$ to be invariant under infinitesimal diffeomorphisms of $M$, 

$$\delta x^\mu = \nu^\mu(x).$$

The condition for this is conservation of the stress tensor

$$D^\mu \langle T_{\mu\nu}(x) \rangle = 0.$$
There is a possible problem: when we compute $\langle T_{\mu\nu}(x) \rangle$ we might run into perturbative anomalies. Specifically, we would like $Z_{\psi}$ to be invariant under infinitesimal diffeomorphisms of $M$, $\delta x^\mu = \nu^\mu(x)$. The condition for this is conservation of the stress tensor

$$D^\mu \langle T_{\mu\nu}(x) \rangle = 0.$$ 

In some theories, we cannot regularize $\langle T_{\mu\nu}(x) \rangle$ in such a way that this condition will be satisfied.
There is a possible problem: when we compute $\langle T_{\mu\nu}(x) \rangle$ we might run into perturbative anomalies. Specifically, we would like $Z_\psi$ to be invariant under infinitesimal diffeomorphisms of $M$, 

$\delta x^\mu = \nu^\mu(x)$. The condition for this is conservation of the stress tensor

$$D^\mu \langle T_{\mu\nu}(x) \rangle = 0.$$ 

In some theories, we cannot regularize $\langle T_{\mu\nu}(x) \rangle$ in such a way that this condition will be satisfied. We say that those theories have perturbative gravitational anomalies and we discard them.
There is a possible problem: when we compute $\langle T_{\mu\nu}(x) \rangle$ we might run into perturbative anomalies. Specifically, we would like $Z_\psi$ to be invariant under infinitesimal diffeomorphisms of $M$, $\delta x^\mu = \nu^\mu(x)$. The condition for this is conservation of the stress tensor

$$D^\mu \langle T_{\mu\nu}(x) \rangle = 0.$$

In some theories, we cannot regularize $\langle T_{\mu\nu}(x) \rangle$ in such a way that this condition will be satisfied. We say that those theories have perturbative gravitational anomalies and we discard them. At least in the traditional view, we only study more subtle questions like modular invariance if the perturbative anomalies cancel (after possibly combining together the contributions of a variety of different bose and fermi fields).
If perturbative anomalies are absent, what might still go wrong?
If perturbative anomalies are absent, what might still go wrong? Once we have a satisfactory definition of the right hand side, the formula

$$\frac{\delta}{\delta g_{\mu\nu}(x)} \log Z_\psi = \langle T_{\mu\nu}(x) \rangle$$

defines $Z_\psi$ as a function of the metric – so let us call it $Z_\psi(g_{\mu\nu})$ – up to an overall phase.
If perturbative anomalies are absent, what might still go wrong? Once we have a satisfactory definition of the right hand side, the formula

\[ \delta \frac{\delta}{\delta g^{\mu\nu}(x)} \log Z_\psi = \langle T_{\mu\nu}(x) \rangle \]

defines \( Z_\psi \) as a function of the metric – so let us call it \( Z_\psi(g^{\mu\nu}) \) – up to an overall phase. The indeterminacy is thus

\[ Z_\psi(g) \to e^{i\alpha} Z_\psi(g) \]

with a constant \( \alpha \).
There can indeed be a problem.
There can indeed be a problem. The condition $D^\mu \langle T_{\mu\nu}(x) \rangle = 0$ ensures invariance of $Z_\psi$ under diffeomorphisms that are continuously connected to the identity, but it may not be invariant under “big” diffeomorphisms that are not so connected.
For example, consider a two-torus $\Sigma = T^2$ parametrized by $x, y$ with $0 \leq x, y \leq 1$. 

An example of a "big" diffeomorphism is $\phi : (x, y) \rightarrow (x + y, y)$. This diffeomorphism corresponds to a modular transformation: if $\Sigma$ has a complex structure defined by saying that $z = x + \tau y$ is holomorphic, then $\phi$ maps $\tau$ to $\tau + 1$. So "modular invariance" is the question of whether $Z\psi$ is invariant under "big" diffeomorphisms such as $\phi$. We describe non-invariance by saying that there is a "global anomaly."
For example, consider a two-torus $\Sigma = T^2$ parametrized by $x, y$ with $0 \leq x, y \leq 1$.

An example of a “big” diffeomorphism is

$$\phi : (x, y) \rightarrow (x + y, y).$$
For example, consider a two-torus $\Sigma = T^2$ parametrized by $x, y$ with $0 \leq x, y \leq 1$.

An example of a “big” diffeomorphism is

$$\phi : (x, y) \rightarrow (x + y, y).$$

This diffeomorphism corresponds to a modular transformation:
For example, consider a two-torus $\Sigma = T^2$ parametrized by $x, y$ with $0 \leq x, y \leq 1$.

An example of a “big” diffeomorphism is

\[ \phi : (x, y) \rightarrow (x + y, y). \]

This diffeomorphism corresponds to a modular transformation: if $\Sigma$ has a complex structure defined by saying that

\[ z = x + \tau y \]

is holomorphic, then $\phi$ maps $\tau$ to $\tau + 1$. 
For example, consider a two-torus $\Sigma = T^2$ parametrized by $x, y$ with $0 \leq x, y \leq 1$.

An example of a “big” diffeomorphism is

$$\phi : (x, y) \rightarrow (x + y, y).$$

This diffeomorphism corresponds to a modular transformation: if $\Sigma$ has a complex structure defined by saying that $z = x + \tau y$ is holomorphic, then $\phi$ maps $\tau$ to $\tau + 1$. So “modular invariance” is the question of whether $Z_\psi$ is invariant under “big” diffeomorphisms such as $\phi$. 
For example, consider a two-torus $\Sigma = T^2$ parametrized by $x, y$ with $0 \leq x, y \leq 1$.

![Diagram of a two-torus]

An example of a “big” diffeomorphism is

$$\phi : (x, y) \rightarrow (x + y, y).$$

This diffeomorphism corresponds to a modular transformation: if $\Sigma$ has a complex structure defined by saying that

$$z = x + \tau y$$

is holomorphic, then $\phi$ maps $\tau$ to $\tau + 1$. So “modular invariance” is the question of whether $Z_\psi$ is invariant under “big” diffeomorphisms such as $\phi$. We describe non-invariance by saying that there is a “global anomaly.”
To decide if $Z_\psi$ is $\phi$-invariant amounts to the following.

Suppose that $g$ is some metric tensor on $M$, and suppose that $\phi$ transforms $g$ to $g \phi$. One wants to know if $Z_\psi(g \phi) = Z_\psi(g)$.

In general, this might be wrong, but it will be always true that $\delta g \log Z_\psi(g \phi) = \delta g \log Z_\psi(g)$.

That is because the left and right hand sides both equal $\langle T_{\mu\nu} \rangle$, which is manifestly invariant under all diffeomorphisms, big or small. (This is true even in anomalous theories: gravitational anomalies mean that $\langle T_{\mu\nu} \rangle$ is not conserved, but it is still diffeomorphism invariant.)
To decide if $Z_\psi$ is $\phi$-invariant amounts to the following. Suppose that $g$ is some metric tensor on $M$, and suppose that $\phi$ transforms $g$ to $g^\phi$. 
To decide if $Z_\psi$ is $\phi$-invariant amounts to the following. Suppose that $g$ is some metric tensor on $M$, and suppose that $\phi$ transforms $g$ to $g^\phi$. One wants to know if $Z_\psi(g^\phi)$ equals $Z_\psi(g)$:

$$Z_\psi(g^\phi) \neq Z_\psi(g).$$
To decide if $Z_\psi$ is $\phi$-invariant amounts to the following. Suppose that $g$ is some metric tensor on $M$, and suppose that $\phi$ transforms $g$ to $g^\phi$. One wants to know if $Z_\psi(g^\phi)$ equals $Z_\psi(g)$:

$$Z_\psi(g^\phi) \overset{?}{=} Z_\psi(g).$$

In general, this might be wrong, but it will be always true that

$$\frac{\delta}{\delta g} \log Z_\psi(g^\phi) = \frac{\delta}{\delta g} \log Z_\psi(g).$$

That is because the left and right hand sides both equal $\langle T_{\mu\nu} \rangle$, which is manifestly invariant under all diffeomorphisms, big or small.
To decide if $Z_\psi$ is $\phi$-invariant amounts to the following. Suppose that $g$ is some metric tensor on $M$, and suppose that $\phi$ transforms $g$ to $g^\phi$. One wants to know if $Z_\psi(g^\phi)$ equals $Z_\psi(g)$:

$$Z_\psi(g^\phi) \overset{?}{=} Z_\psi(g).$$

In general, this might be wrong, but it will be always true that

$$\frac{\delta}{\delta g} \log Z_\psi(g^\phi) = \frac{\delta}{\delta g} \log Z_\psi(g).$$

That is because the left and right hand sides both equal $\langle T_{\mu\nu} \rangle$, which is manifestly invariant under all diffeomorphisms, big or small. (This is true even in anomalous theories: gravitational anomalies mean that $\langle T_{\mu\nu} \rangle$ is not conserved, but it is still diffeomorphism invariant.)
It follows that

\[ Z_\psi(g^\psi) = e^{i\alpha} Z_\psi(g) \]

where \( \alpha \) is a constant, independent of the metric, and moreover \( \alpha \) is real, since the absolute value \( |Z_\psi| \) was well-defined to begin with.
It follows that

\[ Z_\psi(g^\phi) = e^{i\alpha} Z_\psi(g) \]

where \( \alpha \) is a constant, independent of the metric, and moreover \( \alpha \) is real, since the absolute value \(|Z_\psi|\) was well-defined to begin with.

The fact that \( \alpha \) does not depend on the metric \( g \) means that it is a topological invariant.
If $\alpha$ is a topological invariant, what is it an invariant of?
If $\alpha$ is a topological invariant, what is it an invariant of? One answer is that it is an invariant of the equivalence class of the “big” diffeomorphism $\phi$, modulo “little” diffeomorphisms (under which $Z_\psi$ is invariant, since by hypothesis there are no perturbative anomalies).
If $\alpha$ is a topological invariant, what is it an invariant of? One answer is that it is an invariant of the equivalence class of the “big” diffeomorphism $\phi$, modulo “little” diffeomorphisms (under which $Z_\psi$ is invariant, since by hypothesis there are no perturbative anomalies). But there is a more convenient answer. If we are studying fermions on a $D$-manifold $M$, we use $\phi$ as gluing data to build a $D + 1$-manifold $Y$. 
We interpolate between the “old” metric $g$ and the new one $g^\phi$ via
\[ g(x; u)_{ij} = (1 - u)g(x)_{ij} + ug^\phi(x)_{ij}, \quad 0 \leq u \leq 1, \quad 1 \leq i, j \leq D \]
and then we define the $D + 1$-dimensional metric
\[ ds^2 = du^2 + g(x; u)_{ij}dx^i dx^j. \]
We interpolate between the “old” metric $g$ and the new one $g^\phi$ via

$$g(x; u)_{ij} = (1 - u)g(x)_{ij} + ug^\phi(x)_{ij}, \quad 0 \leq u \leq 1, \quad 1 \leq i, j \leq D$$

and then we define the $D + 1$-dimensional metric

$$ds^2 = du^2 + g(x; u)_{ij}dx^i dx^j.$$

This can be viewed as a metric on a $D + 1$-manifold.
We interpolate between the “old” metric $g$ and the new one $g^\phi$ via
\[ g(x; u)_{ij} = (1 - u)g(x)_{ij} + ug^\phi(x)_{ij}, \quad 0 \leq u \leq 1, \quad 1 \leq i, j \leq D \]
and then we define the $D + 1$-dimensional metric
\[ ds^2 = du^2 + g(x; u)_{ij}dx^i dx^j. \]
This can be viewed as a metric on a $D + 1$-manifold

which is a product $M \times I$ topologically, but whose metric is not a product.
The two ends of this $D + 1$-manifold are equivalent via the “big” diffeomorphism $\phi$, so we can glue the two ends together to make a compact $D + 1$-manifold $Y$ without boundary that is known as the mapping torus:
The two ends of this $D+1$-manifold are equivalent via the “big” diffeomorphism $\phi$, so we can glue the two ends together to make a compact $D+1$-manifold $Y$ without boundary that is known as the mapping torus:
The two ends of this $D + 1$-manifold are equivalent via the “big” diffeomorphism $\phi$, so we can glue the two ends together to make a compact $D + 1$-manifold $Y$ without boundary that is known as the mapping torus:

![Diagram of a mapping torus]

The best answer to the question “what is the global anomaly a topological invariant of?” is that it is a topological invariant of the mapping torus $Y$. 
I found a formula of sorts for the global anomaly as a topological invariant of the mapping torus \( Y \) in “Global Gravitational Anomalies” (1985).
I found a formula of sorts for the global anomaly as a topological invariant of the mapping torus $Y$ in “Global Gravitational Anomalies” (1985). The strategy was just to go to a limit in which the $u$ dependence of the metric is adiabatic, and to compute using the usual adiabatic approximation of quantum mechanics.
The answer was as follows.
The answer was as follows. One considers a non-chiral Dirac operator on the $D + 1$-manifold $Y$. It has an eigenvalue problem

$$i \slashed{D} \psi = \lambda \psi$$

and an Atiyah-Patodi-Singer $\eta$-invariant

$$\eta = \lim_{s \to 0} \sum_{i} |\lambda_i|^{-s} \text{sign} \lambda_i.$$
The answer was as follows. One considers a non-chiral Dirac operator on the $D + 1$-manifold $Y$. It has an eigenvalue problem

\[ i \mathcal{D} \psi = \lambda \psi \]

and an Atiyah-Patodi-Singer $\eta$-invariant

\[ \eta = \lim_{s \to 0} \sum_i |\lambda_i|^{-s} \text{sign} \, \lambda_i. \]

I showed that this determines the global anomaly:

\[ e^{i \alpha} = e^{i \pi \eta}. \]
The statement that the global anomaly is given by $e^{i \pi \eta}$ may not be very familiar, but it is a slight refinement of something that actually is well-known.
The statement that the global anomaly is given by $e^{i\pi \eta}$ may not be very familiar, but it is a slight refinement of something that actually is well-known. Perturbative anomalies in $D$ dimensions are related to Chern-Simons in $D + 1$-dimensions and thereby to a characteristic class in $D + 2$ dimensions.
The statement that the global anomaly is given by $e^{i\pi \eta}$ may not be very familiar, but it is a slight refinement of something that actually is well-known. *Perturbative* anomalies in $D$ dimensions are related to Chern-Simons in $D + 1$-dimensions and thereby to a characteristic class in $D + 2$ dimensions. According to Atiyah-Patodi-Singer, $\eta \pmod{2}$ differs from Chern-Simons by a constant:

$$\delta \eta = \delta \text{CS}.$$
The statement that the global anomaly is given by $e^{i\pi \eta}$ may not be very familiar, but it is a slight refinement of something that actually is well-known. *Perturbative* anomalies in $D$ dimensions are related to Chern-Simons in $D + 1$-dimensions and thereby to a characteristic class in $D + 2$ dimensions. According to Atiyah-Patodi-Singer, $\eta$ (mod 2) differs from Chern-Simons by a constant:

$$\delta \eta = \delta \text{CS}.$$ 

This means roughly that $\eta$ and Chern-Simons differ by a topological invariant.
The statement that the global anomaly is given by $e^{i\pi\eta}$ may not be very familiar, but it is a slight refinement of something that actually is well-known. Perturbative anomalies in $D$ dimensions are related to Chern-Simons in $D + 1$-dimensions and thereby to a characteristic class in $D + 2$ dimensions. According to Atiyah-Patodi-Singer, $\eta \pmod{2}$ differs from Chern-Simons by a constant:

$$\delta \eta = \delta \text{CS}.$$ 

This means roughly that $\eta$ and Chern-Simons differ by a topological invariant. Chern-Simons describes perturbative anomalies and $\eta$ is a slight refinement of Chern-Simons that describes global anomalies as well.
Generically $e^{i\pi \eta}$ is not a topological invariant, but it becomes a topological invariant precisely in those theories that have no perturbative anomaly.
Generically $e^{i\pi\eta}$ is not a topological invariant, but it becomes a topological invariant precisely in those theories that have no perturbative anomaly. That is basically because the variation of $\eta$ (when one changes the metric or gauge field on $Y$) equals the variation of Chern-Simons, which controls the perturbative anomaly.
Generically $e^{i\pi \eta}$ is not a topological invariant, but it becomes a topological invariant precisely in those theories that have no perturbative anomaly. That is basically because the variation of $\eta$ (when one changes the metric or gauge field on $Y$) equals the variation of Chern-Simons, which controls the perturbative anomaly. The upshot is that a theory is free of global anomalies if and only if $e^{i\pi \eta} = 1$ for every $D + 1$-manifold that is a mapping torus.
Physically, there is more to life than making sure that there are no global anomalies.
Physically, there is more to life than making sure that there are no global anomalies. Saying the path integral is anomaly-free means that the phase of $Z_\psi$ on any given $M$ can be defined without running into a contradiction, but it does not tell us what is the overall phase of $Z_\psi$. 
Physically, there is more to life than making sure that there are no global anomalies. Saying the path integral is anomaly-free means that the phase of $Z_\psi$ on any given $M$ can be defined without running into a contradiction, but it does not tell us what is the overall phase of $Z_\psi$. If we pick overall phases of $Z_\psi$ at random for different $M$’s, we will violate unitarity and gluing.
Physically, there is more to life than making sure that there are no global anomalies. Saying the path integral is anomaly-free means that the phase of $Z_\psi$ on any given $M$ can be defined without running into a contradiction, but it does not tell us what is the overall phase of $Z_\psi$. If we pick overall phases of $Z_\psi$ at random for different $M$’s, we will violate unitarity and gluing. We want a consistent way to define overall phases of $Z_\psi$ for all possible $M$’s.
It is not hard to give examples of theories that do not have global anomalies but are apparently inconsistent because there is no consistent way to define the overall phases of the path integral on different $M$'s.
It is not hard to give examples of theories that do not have global anomalies but are apparently inconsistent because there is no consistent way to define the overall phases of the path integral on different $M$'s. The perturbative heterotic string (in certain backgrounds) is one example, and massless Majorana fermions in three dimensions lead to another example.
Although I do not claim a complete proof, I believe that there is a general answer for when a theory with fermions is completely consistent and anomaly-free, meaning that the path integral on a general manifold can be defined in a way that is anomaly-free and consistent with all principles of unitarity, locality and cutting and pasting.
Although I do not claim a complete proof, I believe that there is a general answer for when a theory with fermions is completely consistent and anomaly-free, meaning that the path integral on a general manifold can be defined in a way that is anomaly-free and consistent with all principles of unitarity, locality and cutting and pasting. The condition is just that

\[ e^{i\pi \eta} = 1 \]

for all \( D + 1 \)-manifolds \( Y \), not just for mapping tori.
Although I do not claim a complete proof, I believe that there is a general answer for when a theory with fermions is completely consistent and anomaly-free, meaning that the path integral on a general manifold can be defined in a way that is anomaly-free and consistent with all principles of unitarity, locality and cutting and pasting. The condition is just that

$$e^{i\pi\eta} = 1$$

for all $D + 1$-manifolds $Y$, not just for mapping tori. Anomaly cancellation gives the same condition just for mapping tori.
As an example, consider the worldsheet path integral of the heterotic string. One integrates over maps $\varphi : \Sigma \to S$, where $\Sigma = M$ is the string worldsheet and $S$ is spacetime.
As an example, consider the worldsheet path integral of the heterotic string. One integrates over maps $\varphi : \Sigma \rightarrow S$, where $\Sigma = M$ is the string worldsheet and $S$ is spacetime. When one analyzes global anomalies, $\varphi$ is extended to a map $\varphi : Y \rightarrow S$, where $Y$ is the mapping torus.
As an example, consider the worldsheet path integral of the heterotic string. One integrates over maps $\varphi : \Sigma \to S$, where $\Sigma = M$ is the string worldsheet and $S$ is spacetime. When one analyzes global anomalies, $\varphi$ is extended to a map $\varphi : Y \to S$, where $Y$ is the mapping torus. One computes $e^{i\pi \eta}$ including contributions from all of the worldsheet fields of the heterotic string.
As an example, consider the worldsheet path integral of the heterotic string. One integrates over maps $\varphi : \Sigma \rightarrow S$, where $\Sigma = M$ is the string worldsheet and $S$ is spacetime. When one analyzes global anomalies, $\varphi$ is extended to a map $\varphi : Y \rightarrow S$, where $Y$ is the mapping torus. One computes $e^{i\pi \eta}$ including contributions from all of the worldsheet fields of the heterotic string. The condition $e^{i\pi \eta} = 1$ has the following interpretation.
Let $T$ be the tangent bundle of spacetime and $V$ the gauge bundle.
Let $T$ be the tangent bundle of spacetime and $V$ the gauge bundle. The $\alpha'$-corrected supergravity equation

$$dH = \text{tr} \ F \wedge F - \text{tr} \ R \wedge R$$

implies that $p_1(T) = p_1(V)$ at the level of differential forms,
Let $T$ be the tangent bundle of spacetime and $V$ the gauge bundle. The $\alpha'$-corrected supergravity equation

$$dH = \text{tr} F \wedge F - \text{tr} R \wedge R$$

implies that $p_1(T) = p_1(V)$ at the level of differential forms, but the condition $e^{i\pi \eta} = 1$ implies the same thing at the level of integral cohomology.
Let $T$ be the tangent bundle of spacetime and $V$ the gauge bundle. The $\alpha'$-corrected supergravity equation

$$dH = \text{tr} F \wedge F - \text{tr} R \wedge R$$

implies that $p_1(T) = p_1(V)$ at the level of differential forms, but the condition $e^{i\pi \eta} = 1$ implies the same thing at the level of integral cohomology. This is more than one can prove via global anomalies alone.
The main evidence for the condition $e^{i\pi\eta} = 1$ is what I call the Dai-Freed theorem (hep-th/9405012).
The main evidence for the condition $e^{i\pi \eta} = 1$ is what I call the Dai-Freed theorem (hep-th/9405012). Dai and Freed stated their result in a way that sounded somewhat abstract to me when I first heard it, and I did not realize that it entailed a better criterion for consistency of theories with fermions.
But now I would describe the Dai-Freed theorem as a useful way to define a fermion path integral.
But now I would describe the Dai-Freed theorem as a useful way to define a fermion path integral. Suppose that we are trying to define a path integral for fermions on $M$, and suppose that $M$ is the boundary of a manifold $X$ (over which any important structures such as spin structures, gauge bundles, etc., are extended).
But now I would describe the Dai-Freed theorem as a useful way to define a fermion path integral. Suppose that we are trying to define a path integral for fermions on $M$, and suppose that $M$ is the boundary of a manifold $X$ (over which any important structures such as spin structures, gauge bundles, etc., are extended).
But now I would describe the Dai-Freed theorem as a useful way to define a fermion path integral. Suppose that we are trying to define a path integral for fermions on $M$, and suppose that $M$ is the boundary of a manifold $X$ (over which any important structures such as spin structures, gauge bundles, etc., are extended).

Given this, the fermion path integral on $M$ can be defined as

$$Z_{\psi}(M) = |Z_{\psi}(M)| \exp(i\pi \eta X).$$
The basic justification for this formula

\[ Z_\psi(M) = |Z_\psi(M)| \exp(i\pi \eta X) \]

for the fermion path integral is: (1) it is gauge-invariant and consistent with unitarity and factorization; (2) it satisfies

\[ \frac{\delta}{\delta g_{\mu\nu}} \log Z_\psi = \langle T_{\mu\nu} \rangle. \]
For this to make sense, we need to know that the choice of $X$ does not matter.
For this to make sense, we need to know that the choice of $X$ does not matter. If we have two $X$’s, say $X_1$ and $X_2$, we can glue $X_1$ to $-X_2$ to make a closed $(D + 1)$-manifold $Y$. 
For this to make sense, we need to know that the choice of $X$ does not matter. If we have two $X$’s, say $X_1$ and $X_2$, we can glue $X_1$ to $-X_2$ to make a closed $(D+1)$-manifold $Y$. 
There is a gluing formula
\[ \exp(i \pi \eta(X_1)) = \exp(i \pi \eta(X_2)) \cdot \exp(i \pi \eta(Y)) \].

We want \( \exp(i \pi \eta(X_1)) = \exp(i \pi \eta(X_2)) \) so that our definition of the fermion path integral is well-defined.

The condition for this is \( \exp(i \pi \eta(Y)) = 1 \) for any \( D+1 \)-manifold \( Y \) without boundary.
There is a gluing formula

\[ \exp(i \pi \eta(X_1)) = \exp(i \pi \eta(X_2)) \cdot \exp(i \pi \eta(Y)). \]
There is a gluing formula

\[ \exp(i\pi\eta(X_1)) = \exp(i\pi\eta(X_2)) \cdot \exp(i\pi\eta(Y)). \]

We want \( \exp(i\pi\eta(X_1)) = \exp(i\pi\eta(X_2)) \) so that our definition of the fermion path integral is well-defined.
There is a gluing formula

\[ \exp(i\pi\eta(X_1)) = \exp(i\pi\eta(X_2)) \cdot \exp(i\pi\eta(Y)). \]

We want \( \exp(i\pi\eta(X_1)) = \exp(i\pi\eta(X_2)) \) so that our definition of the fermion path integral is well-defined. The condition for this is

\[ \exp(i\pi\eta(Y)) = 1 \]

for any \( D+1 \)-manifold \( Y \) without boundary.
So this is the general answer for when a definition of the fermion path integral based on the Dai-Freed theorem makes sense: One wants $\exp(i\pi \eta) = 1$ for any closed $D + 1$-manifold $X$, with no restriction to mapping tori.
What if no suitable $X$ exists?

When this happens, we get a family of possible definitions of the fermion path integral. I will just explain this with an example: If we are in dimension 2, and $M$ has an odd spin structure, then $M$ is not a boundary but two copies of $M$ are a boundary. In this situation, we get no way to define $Z_M$, but we can define $Z^2_M$:

$$Z^2_\psi(M) = |Z_\psi(M)|^2 \exp(i\pi \eta(X)).$$
What if no suitable $X$ exists? When this happens, we get a family of possible definitions of the fermion path integral.
What if no suitable $X$ exists? When this happens, we get a family of possible definitions of the fermion path integral. I will just explain this with an example:

If we are in dimension 2, and $M$ has an odd spin structure, then $M$ is not a boundary but two copies of $M$ are a boundary. In this situation, we get no way to define $Z_M$, but we can define $Z_2M$:

$$Z_2\psi(M) = |Z_\psi(M)|^2 \exp(i\pi \eta(X)).$$
What if no suitable $X$ exists? When this happens, we get a family of possible definitions of the fermion path integral. I will just explain this with an example: If we are in dimension 2, and $M$ has an odd spin structure, then $M$ is not a boundary but two copies of $M$ are a boundary.
What if no suitable $X$ exists? When this happens, we get a family of possible definitions of the fermion path integral. I will just explain this with an example: If we are in dimension 2, and $M$ has an odd spin structure, then $M$ is not a boundary but two copies of $M$ are a boundary.

\[
Z_{2M} = |Z_M|^2 \exp(i\pi \eta(X))
\]
What if no suitable $X$ exists? When this happens, we get a family of possible definitions of the fermion path integral. I will just explain this with an example: If we are in dimension 2, and $M$ has an odd spin structure, then $M$ is not a boundary but two copies of $M$ are a boundary.

In this situation, we get no way to define $Z_M$, but we can define $Z^2_M$:

$$Z_\psi(M)^2 = |Z_\psi(M)|^2 \exp(i\pi \eta(X)).$$
Similarly we can define $Z_\psi(M)Z_\psi(M')$, if $M$ and $M'$ both have odd spin structure. So what is undetermined is an overall sign, independent of $M$, whenever $M$ has odd spin structure. That is the right answer: The theory remains consist if one reverses the sign of $Z_\psi(M)$ whenever $M$ has odd spin structure. I believe that this is the right answer in general: whatever is not determined by the definition of $Z_\psi$ given by the Dai-Freed theorem is really undetermined, and represents a free parameter in the theory. Depending on the context, the free parameter could be regarded as a coupling constant or (in string theory) a background field. (Nevertheless in some problems, one wants a better understanding of the undetermined parameters. In Freed and Moore, hep-th/0409135, a more precise treatment was given in one example – the low energy effective action of M-theory.)
Similarly we can define $Z_\psi(M)Z_\psi(M')$, if $M$ and $M'$ both have odd spin structure. So what is undetermined is an overall sign, independent of $M$, whenever $M$ has odd spin structure.
Similarly we can define $Z_\psi(M)Z_\psi(M')$, if $M$ and $M'$ both have odd spin structure. So what is undetermined is an overall sign, independent of $M$, whenever $M$ has odd spin structure. That is the right answer: The theory remains consist if one reverses the sign of $Z_\psi(M)$ whenever $M$ has odd spin structure.
Similarly we can define $Z_{\psi}(M)Z_{\psi}(M')$, if $M$ and $M'$ both have odd spin structure. So what is undetermined is an overall sign, independent of $M$, whenever $M$ has odd spin structure. That is the right answer: The theory remains consist if one reverses the sign of $Z_{\psi}(M)$ whenever $M$ has odd spin structure. I believe that this is the right answer in general: whatever is not determined by the definition of $Z_{\psi}$ given by the Dai-Freed theorem is really undetermined, and represents a free parameter in the theory.
Similarly we can define $Z_\psi(M)Z_\psi(M')$, if $M$ and $M'$ both have odd spin structure. So what is undetermined is an overall sign, independent of $M$, whenever $M$ has odd spin structure. That is the right answer: The theory remains consist if one reverses the sign of $Z_\psi(M)$ whenever $M$ has odd spin structure. I believe that this is the right answer in general: whatever is not determined by the definition of $Z_\psi$ given by the Dai-Freed theorem is really undetermined, and represents a free parameter in the theory. Depending on the context, the free parameter could be regarded as a coupling constant or (in string theory) a background field. Nevertheless in some problems, one wants a better understanding of the undetermined parameters. In Freed and Moore, hep-th/0409135, a more precise treatment was given in one example – the low energy effective action of M-theory.
Similarly we can define $Z_\psi(M)Z_\psi(M')$, if $M$ and $M'$ both have odd spin structure. So what is undetermined is an overall sign, independent of $M$, whenever $M$ has odd spin structure. That is the right answer: The theory remains consist if one reverses the sign of $Z_\psi(M)$ whenever $M$ has odd spin structure. I believe that this is the right answer in general: whatever is not determined by the definition of $Z_\psi$ given by the Dai-Freed theorem is really undetermined, and represents a free parameter in the theory. Depending on the context, the free parameter could be regarded as a coupling constant or (in string theory) a background field.

(Nevertheless in some problems, one wants a better understanding of the undetermined parameters. In Freed and Moore, hep-th/0409135, a more precise treatment was given in one example – the low energy effective action of M-theory.)
As I've already mentioned, the perturbative heterotic string is one example in which the answer that comes from the Dai-Freed theorem is sharper than what one learns just from anomalies.
As I've already mentioned, the perturbative heterotic string is one example in which the answer that comes from the Dai-Freed theorem is sharper than what one learns just from anomalies. (I treated it this way in hep-th/9907041.)
As I’ve already mentioned, the perturbative heterotic string is one example in which the answer that comes from the Dai-Freed theorem is sharper than what one learns just from anomalies. (I treated it this way in hep-th/9907041.) I want to give a more contemporary example, which will also lead us to reconsider M2-branes and other string/M-theory branes (except that we won’t have time for details).
The example that I want to give is actually very fashionable in current work in condensed matter physics, where it appears in the theory of a (mostly hypothetical) topological superconductor.
The example that I want to give is actually very fashionable in current work in condensed matter physics, where it appears in the theory of a (mostly hypothetical) topological superconductor. The theory we consider is simply a two-component massless Majorana fermion in three dimensions, coupled to gravity only

\[ l = \int d^3x \ i \bar{\psi} \gamma \phi \psi. \]
The example that I want to give is actually very fashionable in current work in condensed matter physics, where it appears in the theory of a (mostly hypothetical) topological superconductor. The theory we consider is simply a two-component massless Majorana fermion in three dimensions, coupled to gravity only

\[ I = \int d^3 x \, i \bar{\psi} \slashed{D} \psi. \]

If we consider this theory on an oriented three-manifold, then its partition function \( Z_M \) is real (because the Dirac operator is hermitian) but not necessarily positive.
The example that I want to give is actually very fashionable in current work in condensed matter physics, where it appears in the theory of a (mostly hypothetical) topological superconductor. The theory we consider is simply a two-component massless Majorana fermion in three dimensions, coupled to gravity only

\[ I = \int d^3x \, i \bar{\psi} \gamma^0 \psi. \]

If we consider this theory on an oriented three-manifold, then its partition function \( Z_M \) is real (because the Dirac operator is hermitian) but not necessarily positive. One can show that there is no anomaly in the sign of \( Z_M \), and therefore one might think the theory is consistent.
The example that I want to give is actually very fashionable in current work in condensed matter physics, where it appears in the theory of a (mostly hypothetical) topological superconductor. The theory we consider is simply a two-component massless Majorana fermion in three dimensions, coupled to gravity only

\[ I = \int d^3 x \ i \bar{\psi} \not{D} \psi. \]

If we consider this theory on an oriented three-manifold, then its partition function \( Z_M \) is real (because the Dirac operator is hermitian) but not necessarily positive. One can show that there is no anomaly in the sign of \( Z_M \), and therefore one might think the theory is consistent. However, even though \( \exp(i \pi \eta) \) is always 1 on an orientable mapping torus, it is not always 1 on an orientable four-manifold, and I believe this means that the theory is inconsistent even if formulated on orientable manifolds only.
The example that I want to give is actually very fashionable in current work in condensed matter physics, where it appears in the theory of a (mostly hypothetical) topological superconductor. The theory we consider is simply a two-component massless Majorana fermion in three dimensions, coupled to gravity only

$$ I = \int d^3x \, i \bar{\psi} \Phi \psi. $$

If we consider this theory on an oriented three-manifold, then its partition function $Z_M$ is real (because the Dirac operator is hermitian) but not necessarily positive. One can show that there is no anomaly in the sign of $Z_M$, and therefore one might think the theory is consistent. However, even though $\exp(i\pi\eta)$ is always 1 on an orientable mapping torus, it is not always 1 on an orientable four-manifold, and I believe this means that the theory is inconsistent even if formulated on orientable manifolds only. There is no natural way to choose the sign of $Z_\psi(M)$, even though it has no anomaly in the traditional sense.
It is more interesting, however, to take advantage of the fact that the theory of the *massless* Majorana fermion is parity-conserving and to try to formulate it on a possibly unorientable three-manifold.
It is more interesting, however, to take advantage of the fact that the theory of the massless Majorana fermion is parity-conserving and to try to formulate it on a possibly unorientable three-manifold. In this case, one finds that there actually is a global anomaly in general, and when we write

$$Z_\psi(M) = |Z_\psi(M)| \exp(i\pi \eta(X))$$

there really is a dependence on $X$. 
It is more interesting, however, to take advantage of the fact that the theory of the *massless* Majorana fermion is parity-conserving and to try to formulate it on a possibly unorientable three-manifold. In this case, one finds that there actually *is* a global anomaly in general, and when we write

\[ Z_\psi(M) = |Z_\psi(M)| \exp(i\pi \eta(X)) \]

there really is a dependence on \( X \). The way condensed matter physicists interpret this is that the massless Majorana fermion cannot exist on a bare three-manifold, but it can exist on a three-manifold that is the boundary of a four-manifold:
The (mostly hypothetical) material that does this is a topological superconductor, which in bulk has a gap for fermionic excitations, but has gapless fermionic modes on the boundary:
The (mostly hypothetical) material that does this is a topological superconductor, which in bulk has a gap for fermionic excitations, but has gapless fermionic modes on the boundary:
The (mostly hypothetical) material that does this is a topological superconductor, which in bulk has a gap for fermionic excitations, but has gapless fermionic modes on the boundary:

When we write

\[ Z_\psi(M) = |Z_\psi(M)| \exp(i\pi \eta(X)), \]

the bulk factor \( \exp(i\pi \eta(X)) \) comes by integrating out the bulk gapped modes that live on \( X \).
The (mostly hypothetical) material that does this is a topological
superconductor, which in bulk has a gap for fermionic excitations,
but has gapless fermionic modes on the boundary:

When we write

$$Z_\psi(M) = |Z_\psi(M)| \exp(i\pi \eta(X)),$$

the bulk factor

$$\exp(i\pi \eta(X))$$

comes by integrating out the bulk gapped modes that live on $X$. 
One can ask more generally (and many condensed matter physicists have asked this) whether a theory of $\nu$ massless Majorana fermions, all transforming the same way under parity, is consistent (on a possibly unorientable three-manifold).
One can ask more generally (and many condensed matter physicists have asked this) whether a theory of $\nu$ massless Majorana fermions, all transforming the same way under parity, is consistent (on a possibly unorientable three-manifold). To answer this question from the present point of view, we need to know whether $\exp(i\pi\nu\eta(X))$ equals 1 for all four-manifolds $X$. It turns out that this is so if and only if $\nu$ is a multiple of 16. $\mathbb{RP}^4$ is an example of a four-manifold with $\exp(i\pi\eta(X)) = \exp(2\pi i/16)$. This number 16 has been discovered and explained from several different points of view in the condensed matter literature, initially by Kitaev. The question as posed by condensed matter physicists is this: For what values of $\nu$ is it possible, by adding interactions, to make the boundary fermions gapped while preserving reflection symmetry? (The relation to $\eta$ was first suggested by Kapustin, Thorngren, Turzillo, Wang, arXiv:1407.7329.)
One can ask more generally (and many condensed matter physicists have asked this) whether a theory of $\nu$ massless Majorana fermions, all transforming the same way under parity, is consistent (on a possibly unorientable three-manifold). To answer this question from the present point of view, we need to know whether $\exp(i\pi\nu\eta(X))$ equals 1 for all four-manifolds $X$. It turns out that this is so if and only if $\nu$ is a multiple of 16.
One can ask more generally (and many condensed matter physicists have asked this) whether a theory of $\nu$ massless Majorana fermions, all transforming the same way under parity, is consistent (on a possibly unorientable three-manifold). To answer this question from the present point of view, we need to know whether $\exp(i\pi \nu \eta(X))$ equals 1 for all four-manifolds $X$. It turns out that this is so if and only if $\nu$ is a multiple of 16. $\mathbb{RP}^4$ is an example of a four-manifold with $\exp(i\pi \eta) = \exp(2\pi i/16)$. 
One can ask more generally (and many condensed matter physicists have asked this) whether a theory of $\nu$ massless Majorana fermions, all transforming the same way under parity, is consistent (on a possibly unorientable three-manifold). To answer this question from the present point of view, we need to know whether $\exp(i\pi \nu \eta(X))$ equals 1 for all four-manifolds $X$. It turns out that this is so if and only if $\nu$ is a multiple of 16. $\mathbb{RP}^4$ is an example of a four-manifold with $\exp(i\pi \eta) = \exp(2\pi i/16)$. This number 16 has been discovered and explained from several different points of view in the condensed matter literature, initially by Kitaev.
One can ask more generally (and many condensed matter physicists have asked this) whether a theory of $\nu$ massless Majorana fermions, all transforming the same way under parity, is consistent (on a possibly unorientable three-manifold). To answer this question from the present point of view, we need to know whether $\exp(i\pi\nu\eta(X))$ equals 1 for all four-manifolds $X$. It turns out that this is so if and only if $\nu$ is a multiple of 16. $\mathbb{RP}^4$ is an example of a four-manifold with $\exp(i\pi\eta) = \exp(2\pi i/16)$. This number 16 has been discovered and explained from several different points of view in the condensed matter literature, initially by Kitaev. The question as posed by condensed matter physicists is this: For what values of $\nu$ is it possible, by adding interactions, to make the boundary fermions gapped while preserving reflection symmetry?
One can ask more generally (and many condensed matter physicists have asked this) whether a theory of $\nu$ massless Majorana fermions, all transforming the same way under parity, is consistent (on a possibly unorientable three-manifold). To answer this question from the present point of view, we need to know whether $\exp(i\pi \nu \eta(X))$ equals 1 for all four-manifolds $X$. It turns out that this is so if and only if $\nu$ is a multiple of 16. $\mathbb{RP}^4$ is an example of a four-manifold with $\exp(i\pi \eta) = \exp(2\pi i/16)$. This number 16 has been discovered and explained from several different points of view in the condensed matter literature, initially by Kitaev. The question as posed by condensed matter physicists is this: For what values of $\nu$ is it possible, by adding interactions, to make the boundary fermions gapped while preserving reflection symmetry? (The relation to $\eta$ was first suggested by Kapustin, Thorngren, Turzillo, Wang, arXiv:1407.7329.)
This is a case in which anomalies do not capture the full picture: an “anomalies only” approach (i.e., only consider $e^{i\pi\eta(Y)}$ where $Y$ is a mapping torus) would tell us that the theory is consistent if $\nu$ is a multiple of 8, but the correct answer is 16.
Actually the case $\nu = 8$ arises in M-theory on the world-volume of an M2-brane.
Actually the case $\nu = 8$ arises in M-theory on the world-volume of an M2-brane. The worldvolume of the M2-brane is a three-manifold that I will call $M$. 
Actually the case $\nu = 8$ arises in M-theory on the world-volume of an M2-brane. The worldvolume of the M2-brane is a three-manifold that I will call $M$. On $M$, there are $\nu = 8$ Majorana fermions, which are coupled to a rank 8 vector bundle (the positive chirality spinors of the normal bundle to $M$ in an eleven-dimensional spacetime $S$).
Actually the case $\nu = 8$ arises in M-theory on the world-volume of an M2-brane. The worldvolume of the M2-brane is a three-manifold that I will call $M$. On $M$, there are $\nu = 8$ Majorana fermions, which are coupled to a rank 8 vector bundle (the positive chirality spinors of the normal bundle to $M$ in an eleven-dimensional spacetime $S$). Since 8 is not a multiple of 16, this theory is inconsistent by itself: in fact it has an anomaly involving the normal bundle though the anomaly is not the full story.
I treated this question assuming that $M$ is orientable and considering anomalies only in hep-th/9609122.
I treated this question assuming that $M$ is orientable and considering anomalies only in hep-th/9609122. A sufficiently accurate answer to deal with this case is as follows: one has to consider not just the fermion Pfaffian $\text{Pf}(\mathcal{D})$ but also the coupling of the M2-brane to the three-form field $C$ of $M$-theory:

$$\text{Pf}(\mathcal{D}) \exp \left( i \int_M C \right).$$

The first factor is anomalous, and the anomaly is canceled by the second factor if this factor also has a suitable anomaly.
To make $\exp \left( i \int_M C \right)$ well-defined, we would ask that the periods of the field strength $G = dC$ should obey Dirac quantization: they should be integral multiples of $2\pi$. 

The shift makes $\exp \left( i \int_M C \right)$ anomalous and this anomaly cancels the anomaly of the fermions.
To make $\exp \left( i \int_{M} C \right)$ well-defined, we would ask that the periods of the field strength $G = dC$ should obey Dirac quantization: they should be integral multiples of $2\pi$. To get an anomaly, we shift the quantization condition on $G$ and require

$$\int_{V} \frac{G}{2\pi} = \frac{1}{2} \int_{V} \frac{p_{1}(T)}{2} \mod \mathbb{Z}.$$ 

Here $p_{1}(T)$ is the first Pontryagin class of the tangent bundle of the spacetime. The integral $\int_{V} p_{1}(T)/2$ is an integer, so the shifted quantization condition says that periods of $G/2\pi$ can be half-integers.
To make $\exp \left( i \int_M C \right)$ well-defined, we would ask that the periods of the field strength $G = dC$ should obey Dirac quantization: they should be integral multiples of $2\pi$. To get an anomaly, we shift the quantization condition on $G$ and require

$$\int_V \frac{G}{2\pi} = \frac{1}{2} \int_V \frac{p_1(T)}{2} \mod \mathbb{Z}. $$

Here $p_1(T)$ is the first Pontryagin class of the tangent bundle of the spacetime. The integral $\int_V p_1(T)/2$ is an integer, so the shifted quantization condition says that periods of $G/2\pi$ can be half-integers. The shift makes $\exp \left( i \int_M C \right)$ anomalous and this anomaly cancels the anomaly of the fermions.
This is almost a sufficient answer if the worldvolume $M$ of the M2-brane is orientable, but $M$-theory does not require this in general (the condition is rather that the normal bundle to $M$ must be orientable).
This is almost a sufficient answer if the worldvolume $M$ of the M2-brane is orientable, but $M$-theory does not require this in general (the condition is rather that the normal bundle to $M$ must be orientable). It is hard with traditional methods to define the worldvolume path integral of the M2-brane properly if $M$ is unorientable, and this case has not been treated in the literature.
This is almost a sufficient answer if the worldvolume $M$ of the M2-brane is orientable, but $M$-theory does not require this in general (the condition is rather that the normal bundle to $M$ must be orientable). It is hard with traditional methods to define the worldvolume path integral of the M2-brane properly if $M$ is unorientable, and this case has not been treated in the literature. However, it can be treated using the Dai-Freed theorem, in a way similar to what is needed to treat the topological superconductor with $\nu = 8$. (The M2-brane problem is a little more complicated because of the coupling to the spinors of the normal bundle.)
What got me into this subject was thinking about a more subtle case that has not been treated in the literature even at the level of anomalies only: The M2-brane path integral for the case of an M2-brane that ends on an M5-brane.
This case is more complicated because the M2-brane fermions live on a three-manifold with boundary.
This case is more complicated because the M2-brane fermions live on a three-manifold with boundary. To analyze this case, one needs to consider the worldvolume path integral of the fermions, the coupling to the $C$-field, and the coupling of the boundary of the M2-brane to the two-form of selfdual curvature that lives on the M5-brane.
This case is more complicated because the M2-brane fermions live on a three-manifold with boundary. To analyze this case, one needs to consider the worldvolume path integral of the fermions, the coupling to the $C$-field, and the coupling of the boundary of the M2-brane to the two-form of selfdual curvature that lives on the M5-brane. All of these factors are anomalous.
This case is more complicated because the M2-brane fermions live on a three-manifold with boundary. To analyze this case, one needs to consider the worldvolume path integral of the fermions, the coupling to the $C$-field, and the coupling of the boundary of the M2-brane to the two-form of selfdual curvature that lives on the M5-brane. All of these factors are anomalous. I believe it is hard to analyze this problem effectively without a tool such as the Dai-Freed theorem.
One can ask what would be a condensed matter analog of the M2-M5 problem.
One can ask what would be a condensed matter analog of the M2-M5 problem. A partial analog would be a topological superconductor with $3+1$-dimensional worldvolume $Y$, whose boundary $M$ is divided in two parts with two different boundary conditions (possibly because half of the boundary is in contact with some other material).
The boundary condition on one part, say $M_1$, is a “free fermion” boundary condition that we discussed before, such that there are $\nu$ massless free fermions on the boundary, and the boundary condition on the other part, $M_2$, is a “gapped symmetry preserving boundary condition,” only possible because of interactions, so that that part of the boundary is gapped. (There is by now an extensive condensed matter literature on such boundary conditions.)
The boundary condition on one part, say $M_1$, is a “free fermion” boundary condition that we discussed before, such that there are $\nu$ massless free fermions on the boundary, and the boundary condition on the other part, $M_2$, is a “gapped symmetry preserving boundary condition,” only possible because of interactions, so that that part of the boundary is gapped. (There is by now an extensive condensed matter literature on such boundary conditions.)
I hope I have at least succeeded today in giving an overview of the tools that are needed to study the subtle fermion integrals that frequently arise in string/M-theory.
I hope I have at least succeeded today in giving an overview of the tools that are needed to study the subtle fermion integrals that frequently arise in string/M-theory. A detailed analysis of a specific problem would really require a different lecture.
I hope I have at least succeeded today in giving an overview of the tools that are needed to study the subtle fermion integrals that frequently arise in string/M-theory. A detailed analysis of a specific problem would really require a different lecture. Write-ups of some of the problems I’ve mentioned – and some similar ones – will appear soon.